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VIBRATION OF A BEAM INDUCED BY HARMONIC MOTION OF A HEAT SOURCE

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In this paper, a solution to the problem of thermally induced vibration of a uniform, simply supported beam is presented. The effect of internal damping on the vibration is considered. The temperature of the rectangular beam changes as a result of heating by a laser beam. The centre of the laser spot moves harmonically around a fixed point of the beam. The exact solution of the problem is obtained by using a Green function method. From the investigation, it can be concluded that if a frequency of the beam vibration is a multiple of the harmonic motion frequency of the heat source, then resonance can occur in the system. A numerical calculation is carried out to illustrate the theory.

1. INTRODUCTION

Thermally induced vibration of beams has practical importance in space vehicles, reactor vessels, turbines and other machine parts, which are subjected to variable heating. Analysis of the behavior of beam structures, which are subjected to heat, have been presented by Boley [1, 2], Yu [3] and Manolis and Beskos [4].

The vibrations of a simply supported, rectangular beam, subjected to a suddenly applied heat input distributed along its span, were analyzed by Boley [1]. Approximate analyses of the effect of damping on the thermally induced motion of beams and plates have been presented by Boley in reference [2]. In another paper [5], Boley and Barber studied the dynamic response of simply supported isotropic beams and plates subjected to rapid heating. Yu [3] extensively explored the problem of thermal flutter of a flexible spacecraft boom. In this work, the author also studied the effect of viscoelastic damping and a viscous fluid damper on the stability of the boom motion. Manolis and Beskos [4] examined thermally induced vibrations of structures composed of beams, which are exposed to rapid surface heating. The problem was then formulated and solved in the Laplace transform domain. Here, the effects of damping and of axial loads on the structural response are also studied. The thermoelastic damping of an isotropic and homogeneous Bernoulli–Euler beam undergoing flexural vibration were considered by Zener [6] and Kinra and Milligan [7]. In reference [8] the vibration of a simply supported beam forced by harmonic motion of lateral force are presented. The vibration problem of a beam and a rectangular plate, with one surface exposed to a moving heat source, has been investigated in reference [9]. In this investigation, the effect of internal damping was neglected.

In this paper, the problem of thermally induced vibration of a uniform, simply supported rectangular beam has been studied. The formulation of the problem takes into consideration the effect of internal damping on the beam vibration. The temperature of the beam changes as a result of heating by a laser beam. The centre of the laser spot moves

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harmonically around a fixed point of the beam. The solution of the problem is obtained analytically by applying a Green function method.

2. PROBLEM FORMULATION

The differential equation of thermally induced lateral vibration of the Bernoulli–Euler beam with the internal damping, can be written in the following form [10].

$$EI\left[\frac{\partial^4 v}{\partial x^4} + f\frac{\partial^5 v}{\partial x^4 \partial t}\right] + \rho\frac{\partial^2 v}{\partial t^2} = \frac{\partial^2 M_T}{\partial x^2}. \tag{1}$$

Here EI is the bending rigidity, ρ is the mass per unit length of the beam, M_T is the thermal moment, v is the lateral beam deflection, f is the internal damping coefficient of the beam material, x is the distance along the length of the beam and t denotes time. The thermal moment is given by

$$M_T(x, t) = \alpha Eh \int_0^h T(x, y, t)\left(y - \frac{h}{2}\right) \mathrm{d}y, \tag{2}$$

where α is the coefficient of thermal expansion, E is Young's modulus, h is the height of the beam and T is the temperature distribution of the beam. The temperature distribution T is a function of x, y and t and satisfies the equation of heat conduction with a thermal diffusivity κ [11]:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{1}{\lambda} g(x, y, t) = \frac{1}{\kappa}\frac{\partial T}{\partial t}. \tag{3}$$

Here the $g(x, y, t)$ term represents volume energy generation and λ is the thermal conductivity. The volume energy generation is assumed in the form:

$$g(x, y, t) = \begin{cases} \dfrac{\theta}{2\varepsilon}\delta(y) & \text{for } \bar{x}(t) - \varepsilon < x < \bar{x}(t) + \varepsilon, \\ 0 & \text{otherwise,} \end{cases} \tag{4}$$

where θ characterizes the stream of heat and $\delta(\cdot)$ is the Dirac delta function. Then $\bar{x}(t)$ describes the movement of the heat source and is assumed as follows (Figure 1):

$$\bar{x}(t) = x_0 + A \sin \varphi t, \tag{5}$$

where $A + \varepsilon < x_0 < L - A - \varepsilon$, and L denotes the length of the beam.

Consider the case of the simply supported beam, for which the initial and boundary conditions are given as follows:

$$v(x, 0) = 0, \qquad \frac{\partial v}{\partial t}(x, 0) = 0, \tag{6}$$

$$v(0, t) = v(L, t) = 0, \qquad \frac{\partial^2 v}{\partial x^2}(0, t) = \frac{\partial^2 v}{\partial x^2}(L, t) = 0. \tag{7}$$

The initial temperature of the beam and the temperature of the beam ends is zero:

$$T(x, y, 0) = 0, \qquad T(0, y, t) = T(L, y, t) = 0. \tag{8, 9}$$

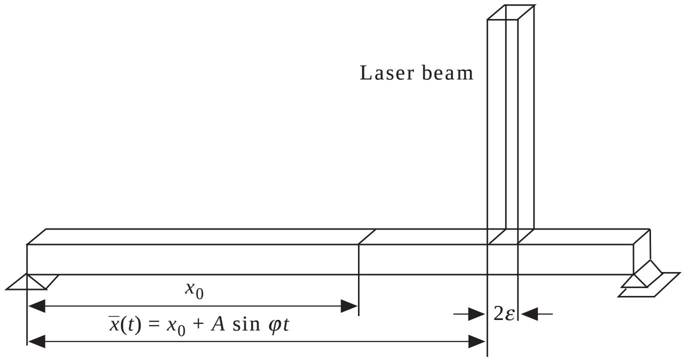

Figure 1. A simply supported beam under an applied heat input.

The convection boundary conditions on the surfaces $y = 0$ and $y = h$ are assumed:

$$\lambda \frac{\partial T}{\partial y}(x, 0, t) = -\alpha_0[T_0 - T(x, 0, t)], \tag{10}$$

$$\lambda \frac{\partial T}{\partial y}(x, h, t) = \alpha_1[T_1 - T(x, h, t)], \tag{11}$$

where α_0 and α_1 are the heat transfer coefficients, and T_0 and T_1 are the known temperatures of the surrounding medium.

3. SOLUTION OF THE PROBLEM

The solution of the problem is obtained by applying a Green function method. The first step is to determine the temperature distribution of the beam. The temperature distribution $T(x, y, t)$ as a solution of the differential problem (3) and (8)–(11), is expressed by the Green function $G(x, y, t; \xi, \eta, \tau)$ as follows:

$$T(x, y, t) = \frac{\kappa}{\lambda} \int_0^t \int_0^L [\alpha_0 T_0 G_T|_{\eta=0} + \alpha_1 T_1 G_T|_{\eta=h}] \,\mathrm{d}\xi \,\mathrm{d}\tau$$

$$+ \frac{\theta\kappa}{2\varepsilon\lambda} \int_0^t \int_{\bar{x}(\tau)-\varepsilon}^{\bar{x}(\tau)+\varepsilon} G_T(x, y, t; \xi, 0, \tau) \,\mathrm{d}\xi \,\mathrm{d}t. \tag{12}$$

The Green function G_T is the solution of the equation

$$\kappa \nabla^2 G_T + \frac{\partial G_T}{\partial \tau} = -\delta(x - \xi)\delta(y - \eta)\delta(t - \tau). \tag{13}$$

Moreover, the function G_T satisfies the initial and homogeneous boundary conditions analogous to conditions (8)–(11). This function can be determined by using the method presented by Carslaw and Jaeger [12]. The function has the form

$$G_T(x, y, t; \xi, \eta, \tau) = \frac{4}{hL} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{q_n^2} \sin\frac{\pi m x}{L} \sin\frac{\pi m \xi}{L} \Psi_n(y)\Psi_n(\eta) \exp(-\gamma_{mn}(t - \tau)), \tag{14}$$

where

$$\Psi_n(y) = \beta_n \cos \beta_n y + \alpha_0 \sin \beta_n y, \qquad q_n^2 = \alpha_0^2 + \beta_n^2 + \frac{\alpha_0 + \alpha_1}{h} \frac{\alpha_0\alpha_1 + \beta_n^2}{\alpha_1^2 + \beta_n^2},$$

$$\gamma_{mn} = \kappa\left[\left(\frac{m\pi}{L}\right)^2 + \beta_n^2\right], \qquad m, n = 1, 2, \ldots$$

and β_n are roots of the equation

$$\alpha_0^2 - \beta_n^2 + (\alpha_0 + \alpha_1)\beta_n \operatorname{ctg} \beta_n h = 0. \tag{15}$$

Substituting the Green function (14) into equation (12) gives

$$T(x, y, t) = \frac{4}{hL} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{q_n^2} D_{mn}^T(t) \sin \frac{\pi m x}{L} \Psi_n(y), \tag{16}$$

where

$$D_{mn}^T(t) = \frac{\kappa}{\lambda\gamma_{mn}} \frac{L}{m\pi} [1 - (-1)^m][\alpha_0\beta_n T_0 + \alpha_1 T_1 \Psi_n(h)][1 - \exp(-\gamma_{mn} t)]$$
$$+ \frac{2\theta\kappa}{\lambda} \beta_n \frac{\sin [m\pi\varepsilon/L]}{m\pi\varepsilon/L} K_{mn}^T(t)$$

and

$$K_{mn}^T(t) = \tfrac{1}{2} \int_0^t \exp(-\gamma_{mn}(t - \tau)) \sin\left[\frac{\pi m x}{L} (x_0 + A \sin \varphi\tau)\right] d\tau. \tag{17}$$

The integral in equation (17) can be calculated by using the following functional relationships [13]:

$$\cos (r \sin u) = 2 \sum_{i=0}^{\infty} \chi_i J_{2i}(r) \cos 2iu, \tag{18}$$

$$\sin (r \sin u) = 2 \sum_{i=0}^{\infty} J_{2i+1}(r) \sin (2i + 1)u, \tag{19}$$

where $\chi_0 = \frac{1}{2}$, $\chi_i = 1$ for $i = 1, 2, \ldots$, $J_v(\cdot)$ denotes the Bessel function of the first kind of order v. After utilizing the relations (18) and (19), equation (17) yields

$$K_{mn}^T(t) = \sin \frac{m\pi x_0}{L} \sum_{i=0}^{\infty} \chi_i J_{2i}\left(\frac{m\pi A}{L}\right) \frac{1}{\gamma_{mn}^2 + 4i^2\varphi^2} U_{imn}^T(t)$$
$$+ \cos \frac{m\pi x_0}{L} \sum_{i=0}^{\infty} J_{2i+1}\left(\frac{m\pi A}{L}\right) \frac{1}{\gamma_{mn}^2 + (2i + 1)^2\varphi^2} W_{imn}^T(t),$$

where

$$U_{imn}^T(t) = 2i\varphi \sin 2i\varphi t + \gamma_{mn}[\cos 2i\varphi t - \exp(-\gamma_{mn} t)]$$

and

$$W^T_{imn}(t) = \gamma_{mn} \sin(2i+1)\varphi t - (2i+1)\varphi[\cos(2i+1)\varphi t - \exp(-\gamma_{mn} t)].$$

The thermal moment is found by substituting the function $T(x, y, t)$, given by equation (16), into equation (2). After calculating the integral, the function M_T is as follows:

$$M_T(x, t) = \alpha E h \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{z_n}{q_n^2} D^T_{mn}(t) \sin \frac{m\pi x}{L}, \tag{20}$$

where

$$z_n = \frac{1}{\beta_n}\left(1 - \frac{\alpha_0 h}{2}\right) \cos \beta_n h + \left(\frac{h}{2} + \frac{\alpha_0}{\beta_n^2}\right) \sin \beta_n h - \frac{1}{\beta_n}\left(1 + \frac{\alpha_0 h}{2}\right).$$

The deflection of the beam is given as a solution of the equation (1) with initial and boundary conditions (6) and (7). The function $v(x, t)$ can be written in the form

$$v(x, t) = \int_0^t \int_0^L \frac{\partial^2 M_T}{\partial \xi^2} G_v(x, t; \xi, \tau)\, d\xi\, d\tau, \tag{21}$$

where G_v denotes the Green function of the differential problem (1) and equations (6) and (7). This function is a solution of the equation

$$EI\left[\frac{\partial^4 G_v}{\partial \xi^4} + f \frac{\partial^5 G_v}{\partial \xi^4\, \partial t}\right] + \rho \frac{\partial^2 G_v}{\partial \tau^2} = \delta(x - \xi)\delta(t - \tau) \tag{22}$$

and satisfies conditions analogous to equations (6) and (7). The function G_v is given as follows:

$$G_v(x, t; \xi, \tau) = \frac{2}{\rho L} \sum_{m=1}^{\infty} a_m(t - \tau) \sin \frac{\pi m x}{L} \sin \frac{\pi m \xi}{L} \sin \omega_m (t - \tau), \tag{23}$$

where

$$\omega_m = \sqrt{\frac{EI}{\rho}} \left(\frac{\pi m}{L}\right)^2, \qquad \Omega_m = \omega_m \sqrt{1 - \left(\frac{f}{2}\omega_m\right)^2}, \qquad \bar{\Omega}_m = \omega_m \sqrt{\left(\frac{f}{2}\omega_m\right)^2 - 1}$$

and

$$a_m(u) = \begin{cases} \dfrac{\sin \Omega_m u}{\Omega_m}, & \text{for } \omega_m f < 2, \\ u, & \text{for } \omega_m f = 2, \\ \dfrac{\sinh \bar{\Omega}_m u}{\bar{\Omega}_m}, & \text{for } \omega_m f > 2. \end{cases}$$

Substituting equations (20) and (23) into equation (21), the function $v(x, t)$ of the beam is expressed as

$$v(x, t) = \alpha h \sqrt{\frac{EI}{\rho}} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{z_n}{q_n^2} D^v_{mn}(t) \sin \frac{\pi m x}{L}, \tag{24}$$

where

$$D^v_{mn}(t) = \frac{1}{\gamma_{mn}}\frac{L}{m\pi}[1-(-1)^m][\alpha_0\beta_n T_0 + \alpha_1 T_1 \Psi_n(h)]P^v_{mn}(t) + \frac{2\theta}{\lambda}\beta_n K^v_{mn}(t), \tag{25}$$

$$K^v_{mn}(t) = \sin\frac{m\pi x_0}{L}\sum_{i=0}^{\infty}\chi_i \mathrm{J}_{2i}\left(\frac{m\pi A}{L}\right)U^v_{imn}(t) + \cos\frac{m\pi x_0}{L}\sum_{i=0}^{\infty}\mathrm{J}_{2i+1}\left(\frac{m\pi A}{L}\right)W^v_{imn}(t). \tag{26}$$

The functions $P^v_{mn}(t)$, $U^v_{imn}(t)$ and $W^v_{imn}(t)$ are calculated as convolutions of the function $g_m(t)$ and functions $(1-\exp(-\gamma_{mn}t))$, $U^T_{imn}(t)$ and $W^T_{imn}(t)$, respectively. These functions for $f=0$ and $\omega_m = k\varphi$, $k \in N$, are given by the following expressions:

$$P^v_{mn}(t) = \frac{1}{\omega_m}(1-\cos\omega_m t) + b_{mn}(t),$$

$$U^v_{imn}(t) = \tfrac{1}{2}t(\gamma_{mn}\sin 2i\varphi - 2i\varphi\cos 2i\varphi t) - \gamma_{mn}b_{mn}(t), \qquad \text{for} \quad \omega_m = 2i\varphi,$$

$$W^v_{imn}(t) = -\tfrac{1}{2}t(\gamma_{mn}\cos(2i+1)\varphi t + (2i+1)\varphi\sin(2i+1)\varphi t) + (2i+1)\varphi b_{mn}(t),$$
$$\text{for} \quad \omega_m = (2i+1)\varphi,$$

where

$$b_{mn}(t) = \frac{1}{\gamma^2_{mn}+\omega^2_m}(\gamma_{mn}\sin\omega_m t - \omega_m\cos\omega_m t + \omega_m\exp(-\gamma_{mn}t)).$$

The functions $U^v_{imn}(t)$ and $W^v_{imn}(t)$ include the terms which cause the increase in the amplitude of the beam vibration.

The functions $P^v_{mn}(t)$, $U^v_{imn}(t)$ and $W^v_{imn}(t)$ in the four remaining cases are given in the Appendix.

4. NUMERICAL RESULTS AND DISCUSSION

The temperature distribution, thermal moment and displacement of the beam are given by equations (16), (20) and (24). From equation (24) it results that the amplitude of beam vibration can increase with time t. The situation appears for $f=0$, if the natural numbers m and k exist such that $\omega_m = k\varphi$. In this case the resonance of the system is observed. For instance, if $\omega_m = 2i\varphi$ for any natural numbers i and m, then in the sum on the right side of equation (24) the following member occurs:

$$v_r(x,t) = \alpha h\frac{\theta}{\lambda}\sqrt{\frac{E}{\rho I}}\, t\sin\frac{m\pi x_0}{L}\mathrm{J}_{\omega_m/\varphi}\left(\frac{m\pi A}{L}\right)\frac{1}{\gamma^2_{mn}+\omega^2_m}(\gamma_{mn}\sin\omega_m t - \omega_m\cos\omega_m t).$$

The member includes the factor that causes the increase of the vibration amplitude with time t.

A numerical calculation is carried out to illustrate the theory. A uniform, simply supported, rectangular beam and heat source which changes position harmonically around the point $x = 0{\cdot}5L$ is considered. The numerical calculations are performed for the following data: $EI = 667{\cdot}0$ [kG m²], $\rho = 3{\cdot}12$ [kg/m], $L = 1{\cdot}0$ [m], $A = 0{\cdot}3$ [m], $T_0 = T_1 = 20{\cdot}0$ [°C], $\kappa = 1{\cdot}29\times 10^{-6}$ [m²/s], $\beta_0 = \beta_1 = 1{\cdot}45$ [1/m]. The numerical values of the temperature and displacements for a short and long time are displayed in Figures 2 and 3.

The time histories of the temperature of the beam surface, $y=0$, in the middle of the beam, $x = 0{\cdot}5L$, for $\varphi = 0{\cdot}2\pi$ and $\theta = 1000{\cdot}0$ [W/m] are presented in Figures 2(a) and 2(b).

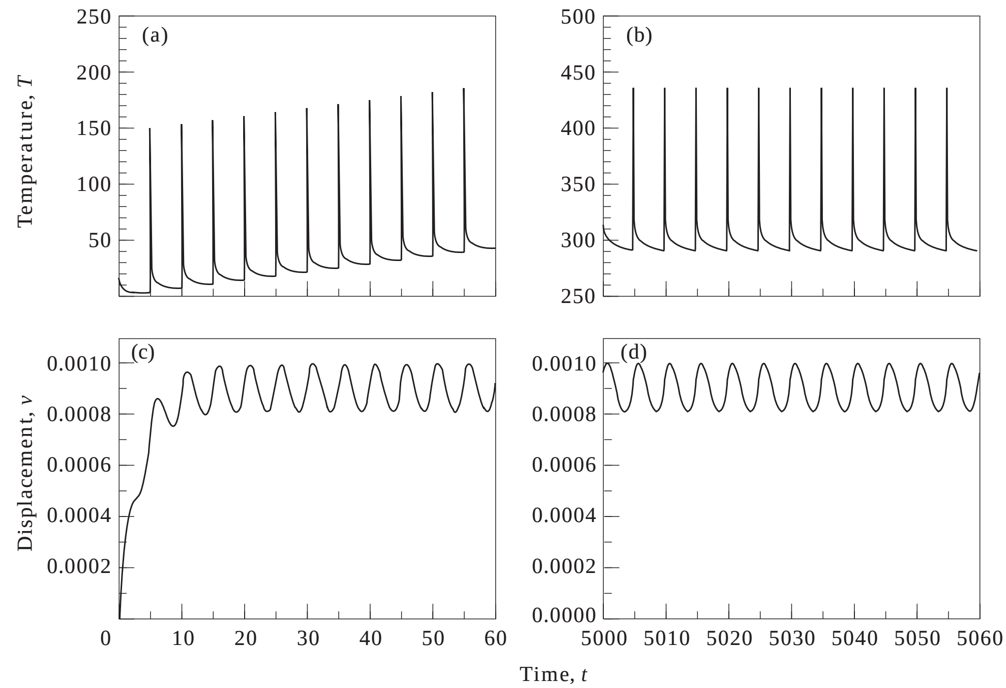

Figure 2. Time histories of the temperature of the beam surface at $x = 0{\cdot}5L$ and displacements of the mid-point of the beam for $A = 0{\cdot}3L$ and $\varphi = 0{\cdot}2\pi$ s^{-1}.

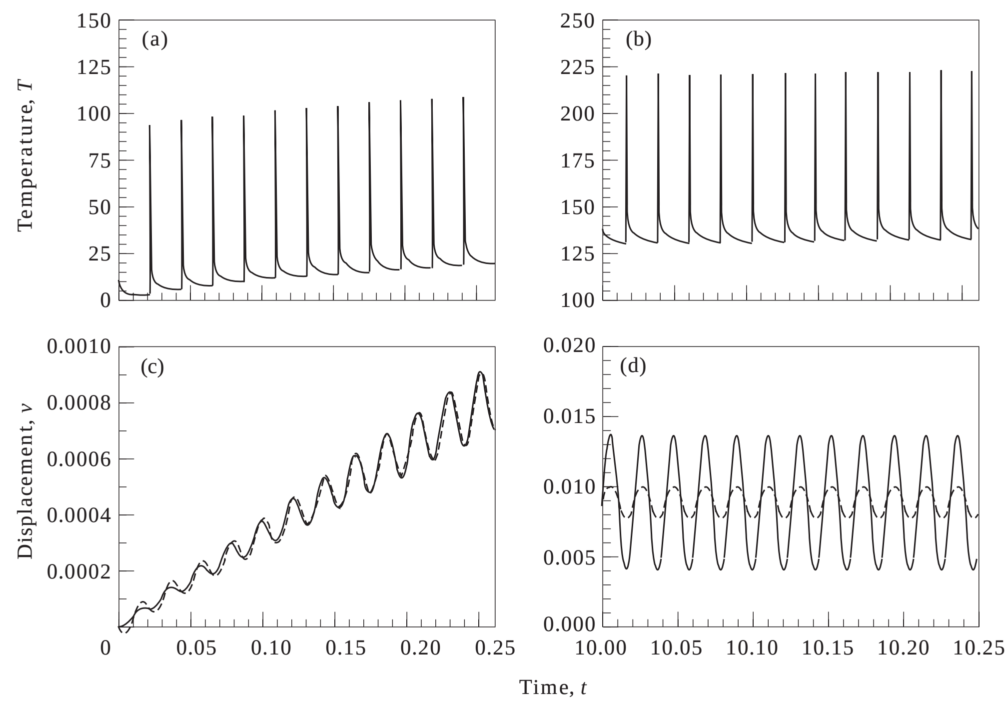

Figure 3. Time histories of the temperature of the beam surface at $x = 0{\cdot}5L$ and displacements of the mid-point of the beam for $A = 0{\cdot}3L$, $\varphi = 0{\cdot}5(\pi/L)^2\sqrt{EI/\rho}$. ——, $f = 0$; – – – –, $f = 10^{-5}$.

The beginning of the process is shown in Figure 2(a) and the results for a long time are presented in Figure 2(b). In Figures 2(c) and 2(d) is shown the displacement of the mid-point of the beam during the vibrations, with the same data as in Figures 2(a) and 2(b), respectively. In this case the periodic vibrations of the beam are induced by the heat source. The increase of the temperature in a long period of time is observed and the vibration of the beam reaches a steady state soon after the beginning of the process. The results of the displacements for the coefficient of internal damping of the beam material, $f = 0$, are shown.

The curves of the temperature and displacement of the mid-point of the beam for $\varphi = 0{\cdot}5\omega_1 = 0{\cdot}5(\pi/L)^2$, $\theta = 10\,000{\cdot}0$ [W/m] are presented in Figure 3. In this case the condition $\omega_1 = 2\varphi$ is satisfied, and as a result the resonance of the system is observed (Figures 3(c) and 3(d)). In Figures 3(b) and 3(d) are shown the histories of the temperature and displacement for a long time. The displacements of the beam mid-point for the internal damping coefficient $f = 0$ (solid line) and $f = 10^{-5}$ (dashed line) are shown in Figures 3(c) and 3(d). The value of the internal damping coefficient for metals is usually taken between 0·005 and 0·02 [4]. However, to show graphically the effect of the internal damping on thermally induced beam vibration, the value $f = 10^{-5}$ is assumed.

5. CONCLUSIONS

A study has been carried out into the thermally induced vibrations of a simply supported Bernoulli–Euler beam. The temperature of the beam is changed by the activity of a heat source which moves harmonically around a fixed point of the beam. The temperature distribution and transverse displacement of the beam, in an analytic form, was obtained by using a Green function method.

It was confirmed that if one frequency of the beam vibration is a multiple of the harmonic motion frequency of the heat source, then resonance can occur in the system. From the numerical investigations it results that, in the non-resonant case, the periodic vibrations of the beam follow immediately after the beginning of the process. In the resonance case, the increase of the vibration amplitude occurs continually during the action.

The presented formulation and solution of the vibration problem takes into consideration the internal damping of the beam material. The effect of the internal damping on the amplitude of the beam vibration is significant in the resonance case, in which a considerable decrease in the beam vibration amplitude is observed.

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APPENDIX

The functions $P^v_{mn}(t)$, $U^v_{imn}(t)$, $W^v_{imn}(t)$, that occur in equations (24) and (25) are given below. Four cases are considered

Case I: $\omega_m f < 2$, $f \neq 0$:

$$P^v_{mn}(t) = \left[A_1 \cos \Omega_m t + A_2 \frac{\sin \Omega_m t}{\Omega_m}\right] \exp(-\tfrac{1}{2} f\omega_m^2 t) + A_3 \exp(-\gamma_{mn} t) + A_4, \tag{A1}$$

$$U^v_{imn}(t) = B_1 \cos 2i\varphi t + B_2 \sin 2i\varphi t + \left[B_3 \cos \Omega_m t + B_4 \frac{\sin \Omega_m t}{\Omega_m}\right] \exp(-\tfrac{1}{2} f\omega_m^2 t)$$
$$+ B_5 \exp(-\gamma_{mn} t), \tag{A2}$$

$$W^v_{imn}(t) = C_1 \cos (2i+1)\varphi t + C_2 \sin (2i+1)\varphi t + \left[C_3 \cos \Omega_m t + C_4 \frac{\sin \Omega_m t}{\Omega_m t}\right]$$
$$\times \exp(-\tfrac{1}{2} f\omega_m^2 t) + C_5 \exp(-\gamma_{mn} t). \tag{A3}$$

Case II: $\omega_m f = 2$, $f \neq 0$:

$$P^v_{mn}(t) = \exp(-2t/f)(A_1 + A_2 t) + \exp(-\gamma_{mn} t)A_3 + A_4, \tag{A4}$$

$$U^v_{imn}(t) = B_1 \cos 2i\varphi t + B_2 \sin 2i\varphi t + \exp(-2t/f)(B_3 + B_4 t) + \exp(-\gamma_{mn} t)B_5, \tag{A5}$$

$$W^v_{imn}(t) = C_1 \cos (2i+1)\varphi t + C_2 \sin (2i+1)\varphi t + \exp(-2t/f)(C_3 + C_4 t)$$
$$+ \exp(-\gamma_{mn} t)C_5. \tag{A6}$$

Case III: $\omega_m f > 2$, $f \neq 0$. Here the functions $P^v_{mn}(t)$, $U^v_{imn}(t)$ and $W^v_{imn}(t)$ can be obtained from the equations of case I. The expressions of the terms in these equations should be changed by replacing $\cos \Omega_m t$ and $\sin \Omega_m t/\Omega_m$ with $\cosh \bar{\Omega}_m t$ and $\sinh \bar{\Omega}_m t/\bar{\Omega}_m$, respectively.

Case IV: $\omega_m \neq k\varphi$, $f = 0$. In this event the equations of case I are applied. The coefficients A, B, C and D in equations (A1)–(A6) are as follows:

$$A_1 = -\frac{\gamma_{mn}(\gamma_{mn} - f\omega_m^2)}{\omega_m D_1}, \qquad A_2 = -\frac{\gamma_{mn}\omega_m}{D_1}[1 + 0{\cdot}5f(\gamma_{mn} - f\omega_m^2)],$$
$$A_3 = -\frac{\omega_m}{D_1}, \qquad A_4 = \frac{1}{\omega_m},$$

$$B_1 = \frac{\omega_m}{D_2}[\gamma_{mn}(\omega_m^2 - 4i^2\varphi^2) - 4f\omega_m^2 i^2\varphi^2], \qquad B_2 = \frac{2i\varphi\omega_m}{D_2}[\omega_m^2 - 4i^2\varphi^2 - f\omega_m^2\gamma_{mn}],$$

$$B_3 = \frac{\omega_m}{D_1 D_2}(\gamma_{mn}^2 + 4i^2\varphi^2)[\gamma_{mn}(4i^2\varphi^2 - \omega_m^2) + f\omega_m^4],$$

$$B_4 = \frac{\omega_m}{D_1 D_2}(\gamma_{mn}^2 + 4i^2\varphi^2)[\omega_n^2(4i^2\varphi^2 - \omega_m^2) - 0{\cdot}5f\gamma_{mn}\omega_m^2 + 0{\cdot}5f^2\omega_m^6], \qquad B_5 = -\frac{\gamma_{mn}\omega_m}{D_1},$$

$$C_1 = -\frac{(2i+1)\varphi\omega_m}{D_3}[\omega_m^2 - (2i+1)^2\varphi^2 + \gamma_{mn}f\omega_m^2],$$

$$C_2 = \frac{\omega_m}{D_3}[\gamma_{mn}(\omega_m^2 - (2i+1)^2\varphi^2) - (2i+1)^2\varphi^2 f\omega^2],$$

$$C_3 = \frac{(2i+1)\varphi\omega_m}{D_1 D_3}(\gamma_{mn}^2 + (2i+1)^2\varphi^2)[\omega_m^2 - (2i+1)^2\varphi^2 + \gamma_{mn}f\omega_m^2 - f^2\omega_m^4],$$

$$C_4 = -\frac{(2i+1)\varphi\omega_m}{D_1 D_3}(\gamma_{mn}^2 + (2i+1)^2\varphi^2)[\gamma_{mn}(\omega_m^2 - (2i+1)^2\varphi^2)$$
$$+ 0{\cdot}5f\omega_m^2(f^2\omega_m^4 + (2i+1)^2\varphi^2) - 0{\cdot}5f\omega_m^4(f\gamma_{mn} + 3\omega_m^2)],$$

$$C_5 = \frac{(2i+1)\varphi\omega_m}{D_1}$$

$$D_1 = \gamma_{mn}^2 + \omega_m^2 - \gamma_{mn}f\omega_m^2, \qquad D_2 = (4i^2\varphi^2 - \omega^2)^2 + 4i^2\varphi^2 f^2\omega_m^4,$$
$$D_3 = ((2i+1)^2\varphi^2 - \omega_m^2)^2 + (2i+1)^2\varphi^2 f^2\omega_m^4.$$